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Sigma-model symmetry in orientifold models

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Abstract: We investigate in the simplest compact $D = 4$ $N = 1$ type-IIB orientifold models the sigma-model symmetry suggested by the proposed duality of these models to heterotic orbifold vacua. This symmetry is known to be present at the classical level, and is associated to a composite connection involving untwisted moduli in the low-energy supergravity theory. In order to study possible anomalies arising at the quantum level, we compute potentially anomalous one-loop amplitudes involving gluons, gravitons and composite connections. We argue that the effective vertex operator associated to the composite connection has the same form as that for a geometric deformation of the orbifold. Assuming this, we are able to compute the complete anomaly polynomial, and find that all the anomalies are canceled through a Green-Schwarz mechanism mediated by twisted RR axions, as previously conjectured. Some questions about the field theory interpretation of our results remain open.

Keywords: Superstring Vacua, Anomalies in Field and String Theories.

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Contents

1. Introduction

Recently, renewed interest has been devoted to orientifold vacua of type-IIB string theory, constructed by projecting out a standard toroidal compactification by the combined action of a discrete spacetime orbifold symmetry G and the world-sheet parity Ω [1]–[4]. These unoriented string theories contain both open and closed strings, and constitute the perhaps most important and concrete example of models in which gauge interactions are localized on D-branes [5]. They are therefore the natural arena for the realization of the "brane-world" scenario. Furthermore, this kind of models have proven to offer surprisingly attractive possibilities from a phenomenological point of view (see for instance [6, 7]).

In the following, we will be concerned with compact $D = 4$ $N = 1$ type-IIB orientifold models [8]–[12]. These vacua represent a simple and tractable prototype of more general and possibly non-supersymmetric orientifold models. Some of them are also phenomenologically appealing and constitute a viable alternative to their more traditional heterotic analogues. In fact, a weak-weak type-IIB heterotic duality has been conjectured [8, 10, 12, 13] for several pairs of vacua.[1] In particular, $\mathbb{Z}_N$ orientifolds with N odd do contain D9-branes but no D5-branes, and could be dual to the corresponding perturbative $\mathbb{Z}_N$ heterotic orbifold. Models with N even do instead contain both D9-branes and D5-branes, and could be dual to heterotic orbifolds with a perturbative sector corresponding to D9-branes and a non-perturbative instantonic sector corresponding to D5-branes [13].

At the classical level, evidence for the duality is suggested by the almost perfect matching of the low-energy spectra and the fact that the orientifold models seem to possess the same classical symmetries as their heterotic companions [20]. In particular, they both possess a so-called "sigma-model" symmetry,[2] naturally emerging from $N = 1$ supergravity. More precisely, this symmetry consists of $\mathrm{SL}(2,\mathbb{R})$ transformations for the untwisted T^i moduli and the other chiral superfields, implemented as the combination of a Kähler transformation and a reparametrization of the scalar Kähler manifold. On the heterotic side, a discrete $\mathrm{SL}(2,\mathbb{Z})$ subgroup of these transformations is known to correspond to the well-known T-duality symmetry, valid to all orders of string perturbation theory, and is therefore expected to be exact. On the orientifold side, instead, sigma-model transformations do not seem to correspond to any known underlying string symmetry, and it is not clear whether the symmetry is exact. At the quantum level, the comparison becomes much more involved and several subtleties arise. In particular, it has been argued in [21] that the one-loop corrected gauge couplings in orientifold models seem to be incompatible with any duality map (see also [22] for further discussion). There is however an important issue which can be addressed even without knowing the detailed duality map: whether or not the classical sigma-model symmetry is anomalous at the quantum level. The latter continuous symmetry is indeed associated to a composite connection in the low-energy effective supergravity theory, and acts as

[1] Notice that, although type IIB $/\Omega$ = type I [1], this duality is not a trivial consequence of type-I heterotic duality in $D = 10$ [14], because in general Ω and G do not commute (see [15] for a discussion in the $D = 6$ models of [16]–[19]), and therefore type IIB $/\{\Omega, G\} \neq$ type I $/G$.

[2] In the following, we shall often use the abbreviation "sigma" for "sigma-model".

chiral rotations on all the fermions. There are therefore anomalous triangular diagrams involving gluons, gravitons and composite connections, leading in general to a non-vanishing one-loop anomaly. On the heterotic side, this one-loop anomaly is canceled by a universal tree-level Green-Schwarz (GS) mechanism mediated by the dilaton [23, 24], and the appearance of the appropriate GS term has been explicitly checked through a string theory computation [25]. On the orientifold side, it was proposed in [20] that a similar GS mechanism involving both the dilaton and twisted RR axions could cancel the anomalies. This observation was motivated by the factorizability of mixed sigma-gauge anomalies computed from the low-energy spectra. However, it was subsequently argued in [22] that requiring a similar mechanism also for mixed sigma-gravitational anomalies would lead to an apparent contradiction with the known results for gauge-gravitational anomaly cancellation [27]. The question of whether the sigma-model symmetry is anomalous or not in orientifold models is therefore still unclear and of extreme relevance for their duality to heterotic theories. Note however that even if the presence of anomalies would pose serious problems to the duality, it would not be fatal for the consistency of the orientifold models in themselves.[3]

The aim of this paper is to study the cancellation of all possible (pure or mixed) sigma-gauge-gravitational anomalies in orientifold models through a string theory computation. Such an analysis is interesting by itself even beyond the context of type-IIB heterotic duality, since it can provide useful informations about the low-energy effective action. For instance, the GS couplings that will be derived are related by supersymmetry to other couplings in the lagrangian and determine under suitable assumptions the gauge kinetic functions and the Fayet-Iliopoulos terms. For simplicity, the analysis will be restricted to the models with N odd. These are indeed simpler than models with N even for a variety of reasons; in particular, they do not present threshold corrections [21]. The only consistent models with N odd are the $\mathbb{Z}_3$ and the $\mathbb{Z}_7$ models.

We follow the strategy developed in [26, 27] for standard gauge-gravitational anomalies, and compute both the quantum anomaly and the classical inflow in all possible channels. By factorization, it is then possible to extract all the anomalous couplings for D-branes and fixed-points present in each model, and the GS term given by their sum. A major ingredient of our computation is an effective vertex operator for the composite sigma-model connection, which results from a pair of untwisted Kähler moduli. We provide arguments that such a vertex is in fact the same as that

[3]Even in the worse case in which all types of mixed anomalies arise, it is always possible to redefine the conserved currents and energy-momentum tensor in such a way to eliminate mixed gauge or gravitational anomalies, and push all the anomaly in the sigma-model symmetry only. The latter is not fatal, since the emerging longitudinal states are composite and not elementary, and so cannot violate unitarity in higher-loop diagrams as would do longitudinal gluons or gravitons resulting from gauge or gravitational anomalies.

of an "internal graviton" associated to a deformation of the Kähler structure of the orbifold respecting its rigid complex-structure. This suggests that there is a close relation between sigma-model symmetry and invariance under reparametrizations of the internal part of the spacetime manifold. In particular, potential anomalies in these symmetries seem to coincide. Assuming the relation above to be valid and using this common vertex, we are able to compute the complete anomaly polynomial as a function of the gauge, gravitational and sigma-model curvatures. We find that all the anomalies are canceled through a GS mechanism mediated by twisted RR axions only, extending the results of [27] for gauge-gravitational anomalies. The dilaton does not play any role in the anomaly cancellation mechanism, contrarily to what proposed in [20] and in agreement with [22].

The results of our string computation disagree with the field theory analysis of [22] on a crucial sign in the contribution of the twisted modulini to the one-loop sigma-gravitational anomaly. Contrary to what assumed in [22], it seems that these twisted closed string states must have a non-vanishing "effective" modular weight, that is responsible for the full cancellation of all the anomalies. Although we do not have yet a complete understanding of the field theory interpretation of our results and their implications on the low-energy effective action, we believe that they rise some questions about the actual form of the Kähler potential for twisted fields. As far as we know, this potential has not yet been unambiguously determined. The only available proposal about its form is that of [28], and it was indeed assumed in [22]. However, this potential implies vanishing modular weight for twisted fields, in apparent contradiction with our results, at least if one does not include possible tree-level corrections to it induced by the GS mechanism. Whether our string results might be explained by taking into account the GS terms in the potential proposed in [28], or they imply a different form for the Kähler potential of twisted fields, has still to be understood [29].

Independently of their actual field theory explanation, we think that our results provide strong evidence for the occurrence of this cancellation mechanism, generalizing it moreover to all the other types of anomalies, like in particular pure sigma-model anomalies. Unfortunately, although we provide several convincing arguments on the correctness of the effective vertex operator for the composite sigma-model connection, a rigorous proof is missing. Therefore, the only safe statement that we are in position to make is that the associated symmetry is preserved at the one-loop level thanks to a generalized GS mechanism. Whether or not this is really the sigma-model symmetry remains strictly speaking to be proven, although we believe that it is quite unlikely that this is not the case since all the anomalies we compute have precisely the structure expected for sigma-gauge-gravitational anomalies. Notice also that thanks to the alternative interpretation of this vertex as an internal graviton, these canceled anomalies can be inequivocably interpreted as relative to internal reparametrizations. As such, they admit a topological interpretation in terms of equivariant indices of the spin and signature complexes, and it is possible to verify the results obtained through the direct string computation by applying suitable index theorems, as we will see.

The structure of the paper is the following. In section 2, we briefly review the notion of sigma-model symmetry. In section 3, we set up the general strategy of the string computation and propose a possible path-integral derivation of the effective vertex for the composite connection. In section 4 we perform the string computation on the four surfaces appearing at the one-loop order. In section 5, we reproduce the same results from a mathematical point of view as topological indices. In section 6, we discuss in more detail the obtained quantum anomalies and perform the factorization of the classical inflow to get all the RR anomalous couplings and the total GS term. In section 7, we discuss possible field-theory interpretations of our results and their implications. Finally, we give conclusions in section 8. In appendix A, we report useful conventions about ϑ-functions. In appendix B, we discuss the cancellation of anomalies in type-IIB string theory (this completes the analysis in [26]). Finally, appendix C contains some useful details about the string computation.

2. Sigma-model symmetry

In this section, we briefly review some well-known facts about the sigma-model symmetry, and discuss its potential anomalies in $D = 4$, $N = 1$ supergravity models. These general concepts are useful for the considerations that will follow, in particular in section 7.

The scalar manifold $\mathcal{M}$ of any generic $D = 4$ $N = 1$ supergravity model is known to be a Kähler manifold, described by a Kähler potential K. At the classical level the lagrangian presents two distinct symmetries (beside possible local gauge symmetries):

- Kähler symmetry, under which the Kähler potential transforms as[4]

$$\kappa^2 K(\Phi^M, \bar{\Phi}^M) \longrightarrow \kappa^2 K(\Phi^M, \bar{\Phi}^M) + F(\Phi^M) + \bar{F}(\bar{\Phi}^M) \,. \tag{2.1}$$

- Global isometries of $\mathcal{M}$, under which

$$\phi^M \longrightarrow \phi'^M(\phi^N) \,. \tag{2.2}$$

Here Φ^M and ϕ^M denote all the chiral multiplets in the model and their lowest components, $F(\Phi)$ is a generic chiral superfield, and κ^2 is Newton's gravitational constant. The fermions ψ^M in the chiral multiplet Φ^M transform also under (2.1) and (2.2). Correspondingly, the fermionic kinetic terms contain a covariant derivative

[4] The lagrangian is invariant under (2.1) if also the superpotential W transforms as $W \to e^{-F} W$. Since W is irrelevant in the considerations that will follow, we will neglect it.

involving the following "Kähler" and "isometry" connections [30]:

$$A_\mu^{(K)} = -\frac{i}{2}\kappa^2 \left(K_M \partial_\mu \phi^M - K_{\bar M} \partial_\mu \phi^{\bar M} \right) , \tag{2.3}$$

$$A_{\mu N}^{(I)M} = i \left(\Gamma^M_{KN} \partial_\mu \phi^K - \Gamma^{\bar M}_{\bar K \bar N} \partial_\mu \phi^{\bar K} \right) . \tag{2.4}$$

Here $\phi^{\bar M} \equiv \bar\phi^M$, K_M and $K_{\bar M}$ denote the derivative of K with respect to the corresponding fields and Γ is the usual Christoffel connection on the Kähler manifold $\mathcal{M}$. Notice that the above connections are not new fundamental states, but composites of the scalar fields.

At the quantum level, the symmetries associated to (2.1) and (2.2) might be spoiled by triangular one-loop graphs involving as external states the connections (2.3) and (2.4), as well as gluons or gravitons. A direct evaluation of these mixed anomalies is not an easy issue, because of the compositeness of the connections. One can however use indirect arguments that rely on the similarity of the structure of the associated anomalous one-loop amplitudes with that of standard U(1)-gauge and U(1)-gravitational anomalies [23, 24]. We shall briefly review this analogy in the following, focusing on the case in which a single composite connection enters as external state in the anomalous diagram.

The considerations made so far are quite general and apply to any $D = 4$ $N = 1$ model. We specialize now to the low-energy lagrangians arising from the type-IIB orientifolds we want to analyze, i.e. the $\mathbb{Z}_3$ and the $\mathbb{Z}_7$ model (see [8, 9, 10] for more details on these string vacua). The massless closed string spectrum of these models contain the gravitational multiplet, a universal chiral multiplet S, three chiral multiplets T^i corresponding to the (complexified) Kähler deformations of the three internal two-tori,[5] and a given number of chiral multiplets M^α arising from the twisted sectors of the orbifold. The open string spectrum (from D9 branes only in these models) contains vector multiplets and three groups of charged chiral multiplets C^a. In order to distinguish the different coordinates of $\mathcal{M}$, we use the index $M = (i, a, \alpha)$ for T^i, C^a and M^α, respectively. As we will see in next sections, the dilaton field S does not participate at all to the GS mechanism canceling the anomalies, and is inert under any gauge, diffeomorphism or sigma-model transformations.

Up to quadratic order in the charged fields, the total Kähler potential of these orientifolds is believed to be [20]:[6]

$$\kappa^2 K_{tot}(\Phi^M, \bar\Phi^M) = -\ln(S+\bar S) - \sum_{i=1}^{3} \ln(T^i + \bar T^i) + \sum_{i=1}^{3} \delta_i^a \frac{\bar C^a C^a}{T^i + \bar T^i} + \\ + \kappa^2 K^{(M)}(M^\alpha, \bar M^\alpha, T^i, \bar T^i) , \tag{2.5}$$

[5]Actually, additional "off-diagonal" untwisted moduli survive the orientifold projection in the special $\mathbb{Z}_3$ model. We do not consider them here for simplicity, and all the considerations that follow are independent of the presence of these fields.

[6]See also [31] for further considerations on the Kähler potential of $D = 4$ orientifold models.

where $K^{(M)}$ is an unknown potential for the twisted fields M^α. As mentioned in the introduction, the sigma-model symmetry we want to study in these orientifold models is the dual of heterotic T-duality. It acts on the fields T^i, C^a and M^α through the following $\mathrm{SL}(2,\mathbb{R})_i$ transformations (no sum over i, $ad-bc=1$):

$$T^i \longrightarrow \frac{a_i T^i - ib_i}{ic_i T^i + d_i}\,, \tag{2.6}$$

$$C^a \longrightarrow \frac{\delta_i^a}{(ic_i T^i + d_i)} C^a\,, \tag{2.7}$$

$$M^\alpha \longrightarrow M'^\alpha(M^\beta, T^i)\,, \tag{2.8}$$

and similarly for the complex conjugate fields. The transformation (2.7) leaves the corresponding (third) term of the Kähler potential (2.5) invariant and (2.8) is chosen in such way to preserve the last contribution $K^{(M)}$. On the other hand, (2.6) produces a non-trivial transformation of the second term. In total, the complete Kähler potential (2.5) undergoes the following Kähler transformation under (2.6), (2.7) and (2.8):

$$\kappa^2 K_{tot}(\Phi^M, \bar\Phi^M) \longrightarrow \kappa^2 K_{tot}(\Phi^M, \bar\Phi^M) + \lambda^i(T^i) + \bar\lambda^i(\bar T^i)\,, \tag{2.9}$$

with

$$\lambda^i(T^i) = \ln(ic_i T^i + d_i)\,. \tag{2.10}$$

The sigma-model symmetry in question is therefore the combination of an isometry and a Kähler transformation, and potential anomalies will therefore involve both connections (2.3) and (2.4).

In order to be able to derive an explicit formula at least for mixed sigma-gauge/gravitational anomalies, we need to make some extra assumptions on the potential $K^{(M)}$ and the transformations (2.8). We take here the one usually considered in the literature, that indeed holds generically for heterotic models [32]:

$$\begin{aligned} \kappa^2 K^{(M)}(M^\alpha, \bar M^\alpha, T^i, \bar T^i) &= \sum_\alpha \prod_{i=1}^{3} (T^i + \bar T^i)^{n_i^\alpha} \bar M^\alpha M^\alpha + \cdots\,, \\ M^\alpha &\longrightarrow (ic_i T^i + d_i)^{n_i^\alpha} M^\alpha\,, \end{aligned} \tag{2.11}$$

where the dots stand for possible higher order terms in $M^\alpha, \bar M^\alpha$. The numbers n_i^α are the so-called "modular weights" [33] of the fields M^α. It is straightforward to see that for the reparametrizations (2.6), (2.7) and (2.8), and the Kähler transformation (2.9) and (2.10) ($F = \lambda^i$), the total connection $Z_\mu^M \equiv A_\mu^{(K)} + A_\mu^{(I)M}{}_M$ transforms as a U(1) connection:[7]

$$Z_\mu^M \longrightarrow Z_\mu^M + \left(1 + 2n_i^M\right) \partial_\mu \operatorname{Im} \lambda^i\,, \tag{2.12}$$

[7]In deriving (2.12) we assumed that the orbifold limit corresponds to $\langle C^a\rangle = \langle M^\alpha\rangle = 0$. The orbifold limit, however, is generically assumed to be given by $\langle m^\alpha\rangle = 0$, where the scalars m^α belong to the linear multiplets L^α, dual of the chiral multiplets M^α [21]. So we are assuming that at leading order $\langle m^\alpha\rangle = 0$ corresponds to $\langle M^\alpha\rangle = 0$.

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where n_i^α are the coefficients defined in (2.11), $n_i^a = -\delta_i^a$, and $n_i^j = -2\delta_i^j$. The sigma-model symmetry can therefore be viewed as a $\mathrm{U}(1)_i$ symmetry with "modular charge" $Q_i^M = (1+2n_i^M)$. The explicit form of Z^M and its field-strength $G^M = dZ^M$ can be easily evaluated. It is actually convenient to disentangle the modular charges Q_i^M from the connection and define the three connections $Z_{\mu,i}$ and their field-strength $G_{\mu,i}$ so that $Z_\mu^M = \sum_i Q_i^M Z_{i,\mu}$ and $G_{\mu\nu}^M = \sum_i Q_i^M G_{i,\mu\nu}$. One finds:

$$Z_{i,\mu} = \frac{i}{2}\frac{\partial_\mu(t^i - \bar{t}^i)}{t^i + \bar{t}^i}\,, \tag{2.13}$$

$$G_{i,\mu\nu} = 2i\frac{\partial_{[\mu}t^i\partial_{\nu]}\bar{t}^i}{(t^i + \bar{t}^i)^2}\,. \tag{2.14}$$

Sigma-gauge/gravitational anomalies can then be computed by treating them as $\mathrm{U}(1)_i$-gauge/gravitational anomalies (in the following denoted briefly by FFG_i and RRG_i anomalies, respectively). Explicit formulae for the anomaly coefficients can be found for example in [20, eqs. (2.8) and (2.12)].

3. Anomalies in orientifold models

In this section, we will set up the general strategy for studying all types of anomalies in chiral orientifold models, and investigate their cancellation. We will begin by reviewing the main aspects of the approach developed in [26, 27] for standard anomalies (see also [34] for a similar analysis in non-geometric models), and generalize it to sigma-gauge-gravitational anomalies.

To begin, we shall briefly recall some basic but important facts about anomalies for the convenience of the reader. Anomalies in a quantum field theory effective action have to satisfy the Wess-Zumino (WZ) consistency condition. These in turn imply that any anomaly in D dimensions is uniquely characterized by a gauge-invariant and closed $(D+2)$-form I. Using the standard WZ-descent notation:[8] $\mathcal{A} = 2\pi i \int I^{(1)}$. The anomaly polynomial I is a characteristic class of the gauge and tangent bundles, of degree $(D+2)/2$ in the curvature two-forms.

3.1 The strategy

The cancellation of anomalies in string theory is achieved in a very natural and elegant way, and is intimately related to more general consistency requirements, like modular invariance and tadpole cancellation. Possible anomalies arise exclusively from boundaries of the moduli space of one-loop string world-sheets. Moreover, direct computations have shown [35] that the whole tower of massive string states contribute in general to anomalies in such a way that these vanish for consistent models, even if the massless spectrum is generically anomalous on its own. From a

[8]The invariant closed $(D+2)$-form I defines locally a non-invariant Chern-Simons $(D+1)$-form $I^{(0)}$ such that $I = dI^{(0)}$, whose gauge variation then defines a (D)-form $I^{(1)}$ through $\delta I^{(0)} = dI^{(1)}$.

low-energy effective field theory point of view, where massive states are integrated out and only the resulting effective dynamics of the light modes is considered, the total one-loop anomaly is canceled by an exactly opposite anomaly arising in tree-level processes involving the magnetic exchange of tensor fields [36]. This is the celebrated Green-Schwarz (GS) mechanism [36], and is an absolutely crucial ingredient for the existence of consistent supersymmetric chiral gauge theories in higher dimensions.

In the following, we will focus on the CP-odd part of the one-loop effective action, where anomalies arise. For consistent models, the exact string theory computation is expected to yield a vanishing anomaly. However, as discussed above, this is interpreted as a non-trivial GS mechanism of anomaly cancellation in a low-energy effective theory valid at energies $E \ll 1/\sqrt{\alpha'}$. In order to get directly this low-energy approximation, one can take the limit $\alpha' \to 0$ from the beginning, before integrating over the world-sheet moduli. The motivation to pursue this strategy, instead of the more direct full string theory computation, is threefold. First, the required computations simplify dramatically. Furthermore, one gets an improved understanding of the low-energy mechanism of anomaly cancellation. Finally, one can extract important WZ couplings appearing in the effective action by factorization [26, 27].

Consider now orientifold models. The relevant anomalous string diagrams are the annulus (A), the Möbius strip (M) and the Klein bottle (K). These world-sheet surfaces lead to potential divergences due to possible tadpoles for massless particles propagating in the transverse channel. Consequently, they also lead to potential anomalies. In addition, also the torus (T) surface can be anomalous, in the limit under consideration. We will see that there are contributions to the anomaly from this diagram, but they turn out to always cancel among themselves.

The most general situation which is allowed by the property that anomalous amplitudes are boundary terms in moduli space is the following. The A, M and K surfaces are parametrized by a real modulus $t \in [0, \infty]$. The contribution from the boundary at $t \to \infty$ is interpreted as the standard quantum anomaly, whereas the contributions from the other boundary at $t \to 0$ is interpreted as classical inflow of anomaly. The T amplitude is instead parametrized by a complex modulus $\tau \in \mathcal{F}$, where $\mathcal{F}$ is the fundamental domain. Again, the contribution from the component $\partial\mathcal{F}_\infty = [-1/2 + i\infty, 1/2 + i\infty]$ of the boundary $\partial\mathcal{F}$ at infinity is interpreted as the standard quantum anomaly, whereas the contribution from the remaining component $\partial\mathcal{F}_0$ should be associated to the classical inflow of anomaly. Summing up, one would therefore get a quantum anomaly $\mathcal{A} = (A + M + K + T)|_\infty$ and a classical inflow $\mathcal{I} = (A+M+K+T)|_0$. It should be however mentioned that the above interpretation for the T surface involves some conceptual subtleties related to modular invariance, that might mix different contributions. Luckily, we will see that the T amplitude gives a vanishing contribution anyhow: the pieces in the $\partial\mathcal{F}_0$ component cancel pairwise thanks to modular invariance [37], that still holds in the $\alpha' \to 0$ limit, whereas the

$\partial\mathcal{F}_\infty$ component vanishes by itself. Moreover, the A, M and K contributions are topological and independent of the modulus. Correspondingly, $\mathcal{A}$ and $\mathcal{I}$ are identical to each other and cancel.

As last important remark, notice that in four dimensions even in non-planar diagrams the closed string state exchanged in the transverse channel is always on-shell, due to the conservation of momentum. Strictly speacking, this means that the usual argument for the cancellation of anomalies at the string level [35] does not apply in this case, giving further motivation for a detailed analysis.

3.2 Set-up of the computation

The computation of the A, M, K and T amplitudes proceeds along the lines of [26, 27], that we shall briefly review and extend. For the time being, we shall assume that the composite connections (2.13) are described by suitable effective vertex operators, postponing a detailed discussion of this issue to next subsection.

An anomaly of the type discussed above, in the CP-odd part of the effective action, is encoded in a one-loop correlation function in the odd spin-structure on the A, M and K surfaces, and in the odd-even and even-odd spin-structures on the T surface, involving gluons, gravitons and composite connections. Denoting by ρ the modulus of the surface and by $\mathcal{F}$ its integration domain, one has on a given surface and spin-structure

$$\mathcal{A}_{1\cdots n} = \int_{\mathcal{F}} d\rho \langle {V_1}' V_2 \cdots V_n J\rangle \,. \tag{3.1}$$

The insertion of the supercurrent J is due to the existence of a world-sheet gravitino zero-mode; more precisely, $J = T_F + \tilde{T}_F$ in the odd spin-structure on the A, M and K surfaces, and $J = T_F, \tilde{T}_F$ in the odd-even and even-odd spin-structures, respectively on T. The vertex V' is taken in the -1-picture in the odd sector and represents an unphysical particle. Taking the latter to be a longitudinally polarized gluon, graviton or composite connection, one computes the variation of the one-loop effective action under gauge, diffeomorphisms or sigma-model transformations. The remaining vertices V are taken in the 0-picture and represent physical background gluons, gravitons or composite connections. Thanks to world-sheet supersymmetry and the limit $\alpha' \to 0$, one can use effective vertex operators which are simpler to handle.

After some formal manipulations, the correlation function above can be rewritten as boundary terms in moduli space [38, 37]

$$\mathcal{A}_{1\cdots n} = \oint_{\partial\mathcal{F}} d\rho \langle W_1 V_2 \cdots V_n\rangle \,, \tag{3.2}$$

where W is an auxiliary vertex defined out of V' for the unphysical particle. Importantly, the vertices V's contain two tangent fermionic zero-modes, whereas W does not contain any of them. The insertion of W, rather than V, for the unphysical particle representing the gauge variation of the one-loop effective action corresponds to the fact that the anomaly $\mathcal{A}$ is given by the WZ descent of the anomaly polynomial I: $A = 2\pi i \int I^{(1)}$. More precisely, one can show [26, 27] that the latter is obtained simply by substituting back V instead of W, that is

$$I_{1\cdots n} = \oint_{\partial\mathcal{F}} d\rho \langle V_1 V_2 \cdots V_n \rangle \,, \tag{3.3}$$

with the convention of working in two more dimensions and omitting the integration over bosonic zero-modes. Finally, it is possible to define the generating functional of all the possible anomalies by exponentiating one representative vertex for each type of particle and compute the resulting deformed partition function Z'. Finally, the total anomaly polynomial is given just by

$$I = \oint_{\partial\mathcal{F}} d\rho Z' \,. \tag{3.4}$$

3.3 Effective vertices

The fact that one can use effective vertices in the computation of the partition function yielding the anomaly polynomial is due to the $\alpha' \to 0$ limit and to certain special properties of correlation functions in supersymmetric spin-structures like those of relevance here. One way to understand this is to notice that the partition functions to be computed are related to topological indices which are almost insensitive to any continuous parameter deformation. From a more technical point of view, there is always a fermionic zero-mode for each spacetime direction. The corresponding Berezin integral in the partition function yields a vanishing result unless the interaction vertices provide one of each fermionic zero-mode. Infact, products of these fermionic zero modes provide a basis of forms of all degrees in the target spacetime, the Berezin integral selecting the appropriate total degree.

On general grounds, it is expected that the effective vertices depend only on the corresponding curvature. Since these behave as two-forms, they must be contracted with two tangent fermionic zero-modes. Moreover, the vertices must be world-sheet supersymmetric. Finally, thanks to the $\alpha' \to 0$ limit, they cannot contain additional momenta, beside from those defining the curvature. These three basic requirements, together with the index structure of the curvatures and conformal invariance, turn out to severely constrain the effective vertices in each case. For gluons and gravitons, they can be derived in a straightforward way as in [26], but for the composite connections (2.13), the analysis is much more involved since the latter are not fundamental fields but composite of the scalar fields of the theory, and there are therefore no vertex operators directly associated to them. Our main observation is that the field-strengths (2.14) have a quadratic dependence on the untwisted t^i and $\bar{t}^{\bar{i}}$ moduli fluctuations. Correspondingly, suitable amplitudes with the insertion of the vertex operators associated to these scalars should reproduce the insertion of the composite connections (2.13). The untwisted t^i moduli are defined as [10]

$$t^i = e^{-\phi_{10}} g_{i\bar{i}} + i\theta_i\,, \tag{3.5}$$

where ϕ_{10} is the ten-dimensional dilaton, $g_{i\bar{i}}$ is the metric component along the T_i^2 torus and θ_i is a RR axion. The real part of these moduli is therefore represented by a NSNS vertex operator, whereas the imaginary part is described by a RR vertex, involving spin-fields and particularly unpleasant to deal with. Notice for the moment that these vertex operators can provide at most one spacetime fermionic zero-mode. Since physical gluons and gravitons bring each two fermionic zero-modes, correlations with an odd number of moduli vanish, as expected from the fact these should come in pairs reconstructing composite connections. Moreover, in the limit of interest, the correlation functions under analysis factorize into an internal correlation among moduli fields and a spacetime correlation among gluons and gravitons.

We now propose an approach to the derivation of the effective vertex for the composite connection, which is not exhaustive but will allow us to emphasize a few important points. Focus for simplicity on a single internal torus only, for which the composite curvature (2.14) becomes (no sum over the indices) $G_{i,\mu\nu} = 2iK_{i\bar{i}}\partial_{[\mu}t^i\partial_{\nu]}\bar{t}^i$, with $K_{i\bar{i}} = (t^i + \bar{t}^i)^{-2}$. On general grounds, one expects the moduli to pair and reconstruct only composite curvatures of this form. At leading order in the momenta, the structure of the internal correlation between two moduli must therefore be as follows:

$$\langle V_{t^i}(p_1)V_{\bar{t}^i}(p_2)\rangle = \alpha_i K_{i\bar{i}} p_{1\mu} t^i p_{2\nu} \bar{t}^i \psi_0^\mu \psi_0^\nu\,, \tag{3.6}$$

$$\langle V_{t^i}V_{t^i}\rangle = \langle V_{\bar{t}^i}V_{\bar{t}^i}\rangle = 0\,, \tag{3.7}$$

where α_i are some coefficients and V_{t^i} and $V_{\bar{t}^i}$ are the vertex operators for the scalars t^i and $\bar{t}^i$. As already mentioned, correlations such as (3.6) are potentially difficult to compute in orientifold models, because the moduli vertices have a simple NSNS real part, but a complicated RR imaginary part. More precisely, the sigma-model curvature can be rewritten as $G_{i\mu\nu} = iK_{i\bar{i}}\partial_{[\mu}(t^i - \bar{t}^i)\partial_{\nu]}(t^i + \bar{t}^i)$, and one has in principle to use one RR vertex $V_{t^i} - V_{\bar{t}^i}$ and one NSNS vertex $V_{t^i} + V_{\bar{t}^i}$. One could then proceed by contracting the NSNS and RR vertex, take the $\alpha' \to 0$ limit and try to figure out which is the effective vertex that, inserted in the correlation function, gives the same result. This procedure is however complicated, so we prefer to use a trick that will allow us to deduce the effective vertex in a quicker (although not rigorous) way.

The point is that correlations involving only pairs of $V_{t^i} + V_{\bar{t}^i}$ vertices are formally proportional to the corresponding correlations involving pairs of $V_{t^i} + V_{\bar{t}^i}$ and $V_{t^i} - V_{\bar{t}^i}$ vertices. Indeed, using (3.6) and (3.7), one gets:

$$\langle (V_{t^i} \pm V_{\bar{t}^i})(p_1)(V_{t^i} + V_{\bar{t}^i})(p_2)\rangle = \alpha_i K_{i\bar{i}}\left(p_{1\mu}t^i p_{2\nu}\bar{t}^i \pm (1 \leftrightarrow 2)\right)\psi_0^\mu\psi_0^\nu\,. \tag{3.8}$$

Due to the symmetrization in $1 \leftrightarrow 2$, one gets a vanishing result for two NSNS vertices (upper sign), but a non-vanishing one for one RR and one NSNS vertices (lower sign). Nevertheless, both of them encode the same non-vanishing coupling α_i, and by careful inspection it is possible to extract the latter also from the vanishing correlation involving only NSNS vertices, after having recognized the zero corresponding to the unavoidable symmetrization. A convenient way to properly remove the zero is to flip the crucial sign by hand in the final result, reconstructing the sigma-model curvature. A similar analysis goes through for correlations involving more than two moduli. Indeed, as will now become clear, the moduli vertices do indeed always contract in pairs associated to composite curvatures, and all of them can be represented by the NSNS real part, keeping track of the zeroes arising by symmetrization.

We are now in position to attempt a derivation of the effective vertex operator for the composite connections (2.13), by considering a correlation involving an even number of moduli real parts and using the trick discussed above. The corresponding NSNS vertex operator can be easily deduced from (3.5), and is given by

$$V_{t^i} + V_{\bar{t}^i} = (t^i + \bar{t}^i) \int d^2 z (\partial X^i + ip \cdot \psi \psi^i)(\bar{\partial} \bar{X}^i + ip \cdot \tilde{\psi} \bar{\tilde{\psi}}^i) e^{ip\cdot X} + \text{c.c.}\,, \tag{3.9}$$

where c.c. stands for complex conjugate.[9] This vertex can be further simplified case by case thanks to the limit $\alpha' \to 0$, and to the presence of fermionic zero-modes. But contrarily to the simpler case of gluons and gravitons, it might happen that pieces of the vertex which are apparently subleading for small momenta, give nevertheless a leading contribution when contracted. We proceed separately for the A, M, K and the T surfaces.

A, M and K surfaces. In this case, one can start with the following effective vertex:

$$V_{\bar{t}^i} + V_{t^i} = ip \cdot \psi_0 (t^i + \bar{t}^i) \int d^2 z \Big[\psi^i \bar{\partial} \bar{X}^i + \tilde{\psi}^i \partial \bar{X}^i + \bar{\psi}^i \bar{\partial} X^i + \bar{\tilde{\psi}}^i \partial X^i + \cdots \Big]\,, \tag{3.10}$$

where the dots represent possibly important fermionic terms, that are difficult to fix unambiguously in the present approach. By exponentiating two of these vertices with momentum $p^i_{1,2}$, and performing a shift on the internal fermions, one gets an effective interaction for the internal bosons. Rescaling then $(X^i, \bar{X}^i) \to g_{i\bar{i}}^{-1/2} (X^i, \bar{X}^i)$ so that the bosonic kinetic terms are normalized, one finds

$$S_{\text{int}} = K_{i\bar{i}} \left(p_{1\mu} t^i p_{2\nu} \bar{t}^i + (1 \leftrightarrow 2) \right) \psi_0^\mu \psi_0^\nu \int d^2 z \Big[\bar{X}^i (\partial + \bar{\partial}) X^i + \cdots \Big]\,. \tag{3.11}$$

[9] Notice that the vertex (3.9) is actually the right one for $g_{i\bar{i}}$, that differs from $t^i + \bar{t}^i$ for a factor $g_S = e^{-\phi_{10}}$. This difference, possibly important for a careful understanding and comparison of string and field theory results (see e.g. footnote 12), is however irrelevant for most of the considerations that will follow. Correspondingly, we effectively identify $g_{i\bar{i}}$ with $t^i + \bar{t}^i$.

As expected, the factor in front of the effective vertex (3.11) has precisely the same form as (3.8) with the + sign, and this interaction term vanishes due to the $1 \leftrightarrow 2$ symmetrization. According to the previous discussion, by flipping the sign of the second term in the brackets, one generates a non-vanishing interaction which can be interpreted as an effective vertex operator for the composite connection. Notice that one would expect such an effective vertex to be world-sheet supersymmetric, whereas the expression obtained above is not. We conclude from this that the expression (3.11) is incomplete, and that additional purely fermionic terms must indeed be present in (3.10) and (3.11). By requiring a world-sheet supersymmetric vertex, it is then easy to deduce the right form for these fermionic terms, and one finds finally

$$V_G^{\text{eff.}} = \frac{1}{2} G_{i,\mu\nu} \psi_0^\mu \psi_0^\nu \int d^2z \left[\bar{X}^i (\partial + \bar{\partial}) X^i + \left(\bar{\psi} - \bar{\tilde{\psi}}\right)^i \left(\psi - \tilde{\psi}\right)^i \right] . \qquad (3.12)$$

T **surface.** In this case, on can effectively take:

$$V_{\bar{t}^i} + V_{t^i} = ip \cdot \psi_0 (t^i + \bar{t}^i) \int d^2z \Big[\psi^i \partial \bar{X}^i + \bar{\psi}^i \partial X^i + \cdots \Big] . \qquad (3.13)$$

The dots represent again possible fermionic terms. By exponentiating and performing a shift on the left-moving internal fermions, one gets an effective interaction for the bosons given by

$$S_{\text{int}} = K_{i\bar{i}} \left(p_{1\mu} t^i p_{2\nu} \bar{t}^i + (1 \leftrightarrow 2) \right) \psi_0^\mu \psi_0^\nu \int d^2z \Big[\bar{X}^i \bar{\partial} X^i + \cdots \Big] . \qquad (3.14)$$

As before, this interaction term vanishes and one has to perform the discussed sign flip to obtain a non-vanishing interaction to be interpreted as an effective vertex operator for the composite connection. Again, since such effective vertex should be world-sheet supersymmetric, we conclude that (3.14) is indeed incomplete, and fix again the missing fermionic terms thanks to world-sheet supersymmetry. Finally, one gets

$$V_G^{\text{eff.}} = \frac{1}{2} G_{i,\mu\nu} \psi_0^\mu \psi_0^\nu \int d^2z \Big[\bar{X}^i \bar{\partial} X^i + \bar{\tilde{\psi}}^i \tilde{\psi}^i \Big] . \qquad (3.15)$$

There is an alternative way to deduce the form of the effective vertices above. Since the NSNS $\mathrm{Re}\, t^i$ scalar is related to the metric of the corresponding internal two-torus, the exponentiation of its vertex induces a geometric deformation of the orbifold along the i-th internal torus. This can be analyzed directly from a σ-model point of view. By doing that, with standard techniques, it is easy to see that the metric deformation associated to the internal T_i^2 torus is represented by (3.12) and (3.15) on the corresponding surfaces, where G_i is now replaced by the geometric curvature of T_i^2. By exploiting the tensorial structure of this curvature, one easily realizes that the components whose derivatives are all along the spacetime directions, like in (3.11), vanish due to a symmetrization, exactly like before. As expected, one is therefore led to use the same trick as above to get a non-vanishing composite field-strength. However, in this way one gets automatically the fermionic terms in (3.12) and (3.15) and also a first clue of the close relation between the field-strength G and the curvature of the internal space. We postpone to section 5 a more precise analysis of this relationship.

Notice also that in heterotic models, where the untwisted moduli consist of NSNS fields only, the correspondence between Kähler deformations of the orbifold and sigma-model symmetry can be unambiguously established. The net result is again that the effective vertex for the composite connection has the same form as that of an internal graviton, like in (3.15). We think that this gives some extra evidence for the relation between sigma-model symmetry and orbifold Kähler deformations also in type-IIB orientifolds. Indeed, although in the latter case the pseudo-scalars $\mathrm{Im}\, t^i$ are RR fields, from a purely geometrical point of view there is no difference with respect to heterotic models, since in both theories $\mathrm{Im}\, t^i$ simply complexifies the geometric Kähler structure of the orbifold/orientifold.

4. String computation

The computation of the partition functions entering the anomaly polynomial closely follow [26, 27]. We proceed separately for the various surfaces. The A, M and K amplitudes are generalizations of the results of [26, 27] to a non-trivial "composite" background. The T amplitude was instead irrelevant in [26, 27], as shown in appendix B for the six-dimensional case, and has therefore to be computed in detail.

As already said, we restrict to the simplest $\mathbb{Z}_3$ and $\mathbb{Z}_7$ models, which do not contain D5-branes neither $N = 2$ sub-sectors. In these models, the k-th element of $\mathbb{Z}_N$ is $g^k = (\theta^k, \gamma_k)$, where θ^k is a rotation of angles $2\pi k v_i$ in the internal two-tori $i = 1, 2, 3$, and γ_k is a non-trivial twist matrix, acting on the Chan-Paton bundle. The Chan-Paton representation of the twist is fixed by the tadpole cancellation condition. For future convenience, and in order to get contact with the notation used in the literature, we define

$$C_k = \prod_{i=1}^{3} (2 \sin \pi k v_i)\,, \tag{4.1}$$

and its sign $\epsilon_k = \mathrm{sign} C_k$. For N odd, the tadpole cancellation condition can then be written as

$$\frac{1}{4}\,\mathrm{tr}(\gamma_{2k}) = \frac{C_{2k}}{C_k} = \frac{C_k}{C_{2k}} \tag{4.2}$$

and holds because actually all the quantities in the equality are equal to a sign, namely ϵ_{2k}/ϵ_k which is equal to -1 for $\mathbb{Z}_3$ and $+1$ for $\mathbb{Z}_7$.

Let us define the characteristic classes which will appear in the polynomial associated to generic sigma-gauge-gravitational anomalies. For the gauge bundle, one has the natural $\mathbb{Z}_N$ Chern character, function of the gauge curvature F, defined as a trace over the Chan-Paton representation:

$$\mathrm{ch}_k(F) = \mathrm{tr}\left[\gamma_k e^{iF/2\pi}\right]. \tag{4.3}$$

This factor appears in the anomaly from charged chiral spinors. For the tangent bundle, the relevant characteristic classes are the Roof-genus, G-polynomial and Hirzebruch polynomial, functions of the gravitational curvature R and defined in terms of the skew eigenvalues λ_a of R as:

$$\widehat{A}(R) = \prod_{a=1}^{D/2} \frac{\lambda_a/4\pi}{\sinh \lambda_a/4\pi}, \tag{4.4}$$

$$\widehat{G}(R) = \prod_{a=1}^{D/2} \frac{\lambda_a/4\pi}{\sinh \lambda_a/4\pi} \left(2\sum_{b=1}^{D/2} \cosh \lambda_b/2\pi - 1\right), \tag{4.5}$$

$$\widehat{L}(R) = \prod_{a=1}^{D/2} \frac{\lambda_a/2\pi}{\tanh \lambda_a/2\pi}. \tag{4.6}$$

These factors appear respectively in the anomaly from chiral spinors, chiral Rarita-Schwinger fields, and self-dual tensor fields. We also introduce three new characteristic classes depending on the composite curvature $G = dZ$, defined in terms of the curvatures G_i in the three internal tori as

$$\widehat{A}_k(G) = \prod_{i=1}^{3} \frac{\sin(\pi k v_i)}{\sin(\pi k v_i + G_i/2\pi)}, \tag{4.7}$$

$$\widehat{G}_k(G) = \prod_{i=1}^{3} \frac{\sin(\pi k v_i)}{\sin(\pi k v_i + G_i/2\pi)} \left(2\sum_{j=1}^{3} \cos(2\pi k v_j + G_j/\pi) - 1\right), \tag{4.8}$$

$$\widehat{L}_k(G) = \prod_{i=1}^{3} \frac{\tan(\pi k v_i)}{\tan(\pi k v_i + G_i/\pi)}. \tag{4.9}$$

These characteristic classes will appear in the anomaly from states transforming as chiral spinors, Rarita-Schwinger fields and self-dual tensors with respect to sigma-model transformations.

A last preliminary comment relevant to all the surfaces is the following. Due to the universal six bosonic zero-modes in the four non-compact spacetime directions and the two extra auxiliary dimensions introduced to deal with the WZ descent, the partition functions will always contain a free-particle contribution proportional to ρ^{-3}. Moreover, the curvatures will always appear multiplied by ρ as twists in the partition function. An important simplification occurs using the fact that only the 6-form component of the partition function is relevant for our purposes: one can scale out the above explicit dependences on the modulus ρ. This will be important also for the modular invariance of the string amplitudes yielding the anomaly in the torus surface, as we shall see.

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4.1 A, M and K surfaces

On the A, M and K surfaces, the boundary of moduli space is given by the component $t \to \infty$ encoding the quantum anomaly, minus the component $t \to 0$ encoding the classical GS inflow. The contribution of each surface to the total anomaly polynomial is given by

$$I_\Sigma = \Big(\lim_{t\to\infty} - \lim_{t\to 0} \Big) Z_\Sigma(t) \,. \tag{4.10}$$

The partition functions $Z_\Sigma(t)$ are in the RR odd spin-structure, and their operatorial representation is

$$\begin{aligned} Z_A(t) &= \frac{1}{4N} \sum_{k=0}^{N-1} \mathrm{Tr}_R \left[g^k (-1)^F e^{-tH} \right] , \\ Z_M(t) &= \frac{1}{4N} \sum_{k=0}^{N-1} \mathrm{Tr}_R \left[\Omega g^k (-1)^F e^{-tH} \right] , \\ Z_K(t) &= \frac{1}{8N} \sum_{k=0}^{N-1} \mathrm{Tr}_{RR} \left[\Omega g^k (-1)^{F+\tilde{F}} e^{-tH} \right] . \end{aligned} \tag{4.11}$$

Here $H = H(R,F,G)$ is the hamiltonian associated to the two-dimensional supersymmetric non-linear σ-model in a gauge, gravitational and composite background defined by the effective vertex operators below, with Neumann boundary conditions. Due to supersymmetry, (4.11) are generalized Witten indices in which only massless modes can contribute [39]. Indeed, it can be verified explicitly that massive world-sheet fermionic and bosonic modes exactly cancel. As a consequence, the partition functions (4.11) are independent of t, and (4.10) vanishes, reflecting anomaly cancellation through the GS mechanism.

The background dependence of the action is encoded in the effective vertices for external particles. In the odd spin-structure on the A, M and K surfaces, the sum $Q + \tilde{Q}$ of the left and right world-sheet supersymmetries is preserved, and there are space-time fermionic zero-modes $\psi_0^\mu = \tilde{\psi}_0^\mu$. In the limit $\alpha' \to 0$, we use the following effective vertex operators for gluons, gravitons and composite sigma-model connections:

$$V_F^{\mathrm{eff.}} = F^a \oint d\tau \lambda^a \,, \tag{4.12}$$

$$V_R^{\mathrm{eff.}} = R_{\mu\nu} \int d^2 z \left[X^\mu (\partial + \bar{\partial}) X^\nu + (\psi - \tilde{\psi})^\mu (\psi - \tilde{\psi})^\nu \right] , \tag{4.13}$$

$$V_G^{\mathrm{eff.}} = G_i \int d^2 z \Big[\bar{X}^i (\partial + \bar{\partial}) X^i + (\bar{\psi} - \bar{\tilde{\psi}})^i (\psi - \tilde{\psi})^i \Big] , \tag{4.14}$$

in terms of the curvature two-forms

$$F^a = \frac{1}{2} F^a_{\mu\nu} \psi_0^\mu \psi_0^\nu \,, \qquad R_{\mu\nu} = \frac{1}{2} R_{\mu\nu\rho\sigma} \psi_0^\rho \psi_0^\sigma \,, \qquad G_i = \frac{1}{2} G_{i,\mu\nu} \psi_0^\mu \psi_0^\nu \,. \tag{4.15}$$

It is now straightforward to compute the partition functions (4.11) on the A, M and K surfaces. The composite background modifies only the internal partition functions, whereas the spacetime contribution has only the standard dependence on the gauge and gravitational backgrounds. The spacetime part can be computed exactly as in [40, 41], and one finds the same results as in [26, 27]. The computation of the internal part is also similar to that in [26, 27], the curvature G entering as a twist. Using ζ-function regularization, one finds

$$\begin{aligned} Z_A &= \frac{i}{4N} \sum_{k=1}^{N-1} C_k \widehat{A}_k(G) \, \mathrm{ch}_k^2(F) \widehat{A}(R) \,, \\ Z_M &= -\frac{i}{4N} \sum_{k=1}^{N-1} C_k \widehat{A}_k(G) \, \mathrm{ch}_{2k}(2F) \widehat{A}(R) \,, \\ Z_K &= \frac{i}{16N} \sum_{k=1}^{N-1} C_{2k} \widehat{L}_k(G) \widehat{L}(R) \,, \end{aligned} \tag{4.16}$$

in terms of the characteristic classes defined before. As anticipated, the partition functions (4.16) are independent of the modulus t. Consequently, the quantum anomaly encoded in the $t \to \infty$ boundary, and the classical inflow associated to $t \to 0$ boundary, are precisely opposite to each other and cancel on each of the A, M and T surfaces.

4.2 T surface

On the T surface, the boundary $\partial\mathcal{F}$ of moduli space splits into the component at infinity, $\partial\mathcal{F}_\infty = [-1/2 + i\infty, 1/2 + i\infty]$, minus the remaining component, $\partial\mathcal{F}_0$, and the contribution to the total anomaly polynomial is given by

$$I_T = \frac{1}{2} \left[\left(\oint_{\partial\mathcal{F}_\infty} - \oint_{\partial\mathcal{F}_0} \right) d\tau \, Z_T(\tau) + \left(\oint_{\partial\mathcal{F}_\infty} - \oint_{\partial\mathcal{F}_0} \right) d\bar{\tau} \, Z_T(\bar{\tau}) \right] . \tag{4.17}$$

The quantities

$$Z_T(\tau) = \sum_\alpha (-1)^\alpha Z_T^{S_\alpha}(\tau) \,, \qquad Z_T(\bar{\tau}) = \sum_{\tilde{\alpha}} (-1)^{\tilde{\alpha}} Z_T^{S_{\tilde{\alpha}}}(\bar{\tau}) \,. \tag{4.18}$$

are the total partition functions in the odd-even and even-odd sector, respectively. More precisely, $\alpha = 2, 3, 4$ represent the RR, RNS$_+$ and RNS$_-$ odd-even spin-structures, and similarly $\tilde{\alpha} = 2, 3, 4$ represent the RR, RNS$_+$ and RNS$_-$ even-odd spin-structures. Their operatorial representation is

$$
\begin{aligned}
Z_T^{RR}(\tau) &= \frac{1}{8N}\sum_{k,l=0}^{N-1} \mathrm{Tr}_{RR}^{(l)}\left[g^k(-1)^F\tilde{g}^k e^{-\tau H}e^{-\bar{\tau}\tilde{H}}\right], \\
Z_T^{RNS_+}(\tau) &= \frac{1}{8N}\sum_{k,l=0}^{N-1} \mathrm{Tr}_{RNS}^{(l)}\left[g^k(-1)^F\tilde{g}^k e^{-\tau H}e^{-\bar{\tau}\tilde{H}}\right], \\
Z_T^{RNS_-}(\tau) &= \frac{1}{8N}\sum_{k,l=0}^{N-1} \mathrm{Tr}_{RNS}^{(l)}\left[g^k(-1)^F\tilde{g}^k (-1)^{\tilde{F}} e^{-\tau H}e^{-\bar{\tau}\tilde{H}}\right]. \qquad (4.19)
\end{aligned}
$$

The expression for the even-odd spin-structures is perfectly similar, with left and right movers exchanged. In this case, H and $\tilde{H} = \tilde{H}(R,F,G)$ are the left and right-moving hamiltonians associated to the two-dimensional supersymmetric non-linear σ-model in a gauge, gravitational and composite background defined by the effective vertex operators below. Notice that whereas the even part of the partition functions is influenced by the backgrounds, the odd part remains trivial. This will lead to holomorphic and anti-holomorphic results in the odd-even and even-odd spin structures. Furthermore, only the odd parts of (4.19) are supersymmetric indices, whereas the even parts receive contributions from all the tower of string states and will therefore depend on τ.

Again, the background dependence of the action is encoded in the effective vertices for external particles. In the odd-even spin-structure on the T surface, the left-moving world-sheet supersymmetry Q is preserved, and there are space-time fermionic zero-modes ψ_0^μ. In the limit $\alpha' \to 0$, we use the following effective vertex operators for gravitons and composite connections:

$$
V_R^{\text{eff.}} = R_{\mu\nu}\int d^2z\left[X^\mu\bar{\partial}X^\nu + \tilde{\psi}^\mu\tilde{\psi}^\nu\right], \qquad (4.20)
$$

$$
V_G^{\text{eff.}} = G_i\int d^2z\left[\bar{X}^i\bar{\partial}X^i + \bar{\tilde{\psi}}^i\tilde{\psi}^i\right], \qquad (4.21)
$$

in terms of the curvature two-forms defined in (4.15). It is then easy to evaluate the partition function on the T surface. The gravitational background influences bosons and left-moving fermions, in a similar way to the cases discussed in appendix B. The composite background influences instead only the internal bosons and left-moving fermions. The evaluation of the internal partition functions is very similar to that reported in appendix B for the six-dimensional case of type IIB on $T^4/\mathbb{Z}_N$, the curvature G being responsible for a twist. In total, one gets:

$$
Z_T(R,G,\tau) = \frac{i}{8N}\sum_{\alpha=2}^{4}(-1)^\alpha \sum_{k,l=0}^{N-1} N_{k,l}\prod_{i=1}^{3}\frac{\theta_\alpha\begin{bmatrix} lv_i \\ kv_i\end{bmatrix}(-G_i/\pi^2|\tau)}{\theta_1\begin{bmatrix} lv_i \\ kv_i\end{bmatrix}(-G_i/\pi^2|\tau)} \times \qquad (4.22)
$$

$$
\times\prod_{a=1}^{2}\left[\frac{(ix_a)}{\theta_1(ix_a/\pi|\tau)}\theta_\alpha(ix_a/\pi|\tau)\right]\frac{\eta^3(\tau)}{\theta_\alpha(0|\tau)},
$$

where $x_a = \lambda_a/2\pi$ and $N_{k,l}$ is the number of fixed-points that are at the same time k and l-fixed ($N_{0,0} = 0$). The result for the odd-even spin-structures is the complex conjugate of (4.22).

It is a lengthy but straightforward exercise to show that the partition function (4.22) is modular invariant. Indeed, one gets

$$
\begin{aligned}
Z_T(R,G,\tau+1) &= Z_T(R,G,\tau)\,, \\
Z_T(R,G,-1/\tau) &= \frac{1}{\tau} Z_T(R\tau, G\tau, \tau) = \tau^2 Z_T(R,G,\tau)\,,
\end{aligned}
\tag{4.23}
$$

where the last step in the second equation is valid for the relevant 6-form component of Z_T. Thanks to the modular invariance of $Z_T(\tau)$ and $Z_T(\bar\tau)$, their integral on various components of $\partial\mathcal{F}$ are related to each other. In fact, only the component $\partial\mathcal{F}_\infty$ at infinity can give a non-vanishing contributions, the remaining four pieces of the remaining component $\partial\mathcal{F}_0$ canceling pairwise, as in [37]. The potential contribution from $\partial\mathcal{F}_\infty$ is interpreted as a quantum sigma-gravitational anomaly.

In order to evaluate the contribution from $\partial\mathcal{F}_\infty$, one has to take the limit $\tau_2 \to \infty$ of the partition function. This is easy to take in untwisted sectors, but in twisted sectors one has to pay attention to the range of the twists. For $l \neq 0$, one gets for instance:

$$
\begin{aligned}
&\prod_{i=1}^{3} \frac{\theta_2 \begin{bmatrix} lv_i \\ kv_i \end{bmatrix} (-G_i/\pi^2|\tau)}{\theta_1 \begin{bmatrix} lv_i \\ kv_i \end{bmatrix} (-G_i/\pi^2|\tau)} \longrightarrow -i\epsilon_l\,, \\
&\prod_{i=1}^{3} \frac{\theta_{3,4} \begin{bmatrix} lv_i \\ kv_i \end{bmatrix} (-G_i/\pi^2|\tau)}{\theta_1 \begin{bmatrix} lv_i \\ kv_i \end{bmatrix} (-G_i/\pi^2|\tau)} \longrightarrow \mp i\epsilon_l q^{1/8} \prod_{i=1}^{3} \exp\left(\frac{-i\epsilon_l G_i}{\pi}\right),
\end{aligned}
$$

where the quantity ϵ_k was defined as the sign of C_k in (4.1).[10] The above expressions already show that the anomaly from RR twisted states does not depend on the composite curvature, and therefore trivially vanishes in $D = 4$. On the contrary, the anomaly from RNS twisted states does depend on the curvature G, and is non-vanishing. The corresponding apparently complex internal contribution to the partition function (4.22) turns out to be actually real, and using the fact that only odd powers of G_i are relevant in $D = 4$, one gets:

$$
\mp i\epsilon_l q^{1/8} \prod_{i=1}^{3} \exp\left(\frac{-i\epsilon_l G_i}{\pi}\right) = \mp q^{1/8}(-i)\,\mathrm{ch}(2G)\,.
$$

[10]It arises here as $\epsilon_l = (-1)^{\sum_i \theta_i(lv_i)}$ and $\epsilon_l = 2\sum_i \theta_i(lv_i) - 3$ in terms of the representative $\theta_i(lv_i) = lv_i - \mathrm{int}(lv_i)$ of the twist lv_i in the interval $[0,1]$.

Finally, the total odd-even and even-odd spin structure partition functions (4.18) are found to behave both in the same way in the limit $\tau_2 \to \infty$, giving:

$$
\begin{aligned}
Z_T \longrightarrow & -\frac{i}{16N} \sum_{k=1}^{N-1} C_{2k} \widehat{L}_k(G) \widehat{L}(R) + \\
& + \frac{i}{2N} \sum_{k=1}^{N-1} C_k \left[\widehat{A}_k(G) \widehat{G}(R) + \widehat{G}_k(G) \widehat{A}(R) \right] + \\
& + \frac{i}{2N} \sum_{k=0}^{N-1} \sum_{l=1}^{N-1} N_{k,l}(-i) \, \mathrm{ch}(G) \widehat{A}(R) \,.
\end{aligned} \tag{4.24}
$$

Since this expression is independent of τ, the remaining integral over τ_1 in $\partial\mathcal{F}_\infty$ is trivial, and according to (4.17), this is also the final result for the T contribution to the anomaly. The first line of (4.24) corresponds to the RR untwisted sector, whose contribution precisely cancels that of the Klein bottle in (4.16). The second line corresponds to the RNS/NSR untwisted sectors and encodes the contributions of the gravitino, dilatino and untwisted modulini. Finally, the last line encodes those of twisted RNS/NSR moduli; notice that all the twisted sectors $l = 1, \ldots, N-1$ give the same contribution, since $N_{k,l}$ takes the same value for all $\{k, l\} \neq \{0, 0\}$ for N odd. Actually, one can check that the relevant 6-form component of the result (4.24) vanishes identically. Some useful details in this respect are reported in appendix C. In conclusion, the total anomaly from the T surface exactly cancels:

$$
Z_T \longrightarrow 0 \,. \tag{4.25}
$$

Note that whereas the vanishing of the T amplitude is expected from modular invariance in a full string context, it has to be explicitly checked in the particular $\alpha' \to 0$ limit we consider. Because of the importance of this result and since we are not aware of any similar computation in the literature, we report in appendix B a similar computation of gravitational anomalies on the T surface for type-IIB string theory in $D = 10$ and $D = 6$ on an orbifold.

5. Topological interpretation

Probably it is interesting to point out that all the anomalies considered so far, eqs. (4.16), have a nice topological interpretation in terms of the G-index of the Dirac operator (A and M) and the G-index of the signature complex [42, 43] (K), with $G = \mathbb{Z}_N$ for a $\mathbb{Z}_N$ orbifold (see [44] for a nice introduction and more details on the G-index).[11]

[11] A relation between anomalous couplings and the $\mathbb{Z}_2$ signature complex was already exploited in [41] in the case of smooth manifolds.

The $\mathbb{Z}_N$ group can be thought to act on the whole ten-dimensional spacetime $X = \mathbb{R}^{1,3} \times T^6$, as well as on the gauge bundle. As before, we denote by $g_k = (\theta^k, \gamma_k)$ the k-th element of the complete $\mathbb{Z}_N$ group. Among other things, this will twist the Chern classes appearing in index theorems. The subspace X_k left invariant by the geometric θ^k is $X_k = \oplus_{i=1}^{N_k} \mathbb{R}^{1,3}$, that is N_k copies of spacetime. When restricted to X_k, the tangent bundle of X decomposes into the tangent and normal bundles $\mathcal{T}_k$ and $\mathcal{N}_k$ of X_k in X. Moreover, the normal bundle $\mathcal{N}_k$ further decomposes naturally into three components $\mathcal{N}_k^i$, in which θ acts as $2\pi v_i$ rotations. The cotangent and spin bundles, which will be relevant for spinor and self-dual tensor fields, have a similar decomposition.

The Dirac-G index theorem is then given by (see e.g. [44])

$$\mathrm{index}(D_{g^k}) = \int_{X^G} \frac{\mathrm{ch}(S^+_{\mathcal{T}_k} - S^-_{\mathcal{T}_k})\,\mathrm{ch}_k(S^+_{\mathcal{N}_k} - S^-_{\mathcal{N}_k})\,\mathrm{ch}_k(F)}{\mathrm{ch}_k(\tilde{\mathcal{N}}_k) e(\mathcal{T}_k)}\,\mathrm{Td}(\mathcal{T}_k^C)\,, \tag{5.1}$$

where $S^\pm_{\mathcal{T}_k}$ and $S^\pm_{\mathcal{N}_k}$ are the positive and negative chirality spin bundles lifted from the tangent and normal bundles, and $\tilde{\mathcal{N}}_k = \oplus_i (-)^i \wedge^i \mathcal{N}_k^*$ in terms of the conormal bundle $\mathcal{N}_k^*$. $e(\mathcal{T}_k)$ and $\mathrm{Td}(\mathcal{T}_k^C)$ are the usual Euler and (complexified) Todd classes:

$$\mathrm{Td}(\mathcal{T}_k^C) = \prod_{a=1}^{2} \frac{x_a}{1-e^{-x_a}} \frac{(-x_a)}{1-e^{x_a}}\,, \qquad e(\mathcal{T}_k) = \prod_{a=1}^{2} x_a\,.$$

By expliciting the other terms appearing in (5.1), one gets

$$\begin{aligned}
\mathrm{ch}(S^+_{\mathcal{T}_k} - S^-_{\mathcal{T}_k}) &= \prod_{a=1}^{2} (e^{x_a/2} - e^{-x_a/2})\,, \\
\mathrm{ch}_k(S^+_{\mathcal{N}_k} - S^-_{\mathcal{N}_k}) &= \prod_{i=1}^{3} (e^{x_i/2} e^{i\pi k v_i} - e^{-x_i/2} e^{-i\pi k v_i})\,, \\
\mathrm{ch}_k(\tilde{\mathcal{N}}_k) &= \prod_{i=1}^{3} (1 - e^{x_i} e^{2i\pi k v_i})(1 - e^{-x_i} e^{-2i\pi k v_i})\,,
\end{aligned} \tag{5.2}$$

where x_a and x_i are the eigenvalues of the curvature two-form on $\mathcal{T}_k$ and $\mathcal{N}_k$. $\mathrm{ch}_k(F)$ is precisely the twisted Chern character defined in (4.3), in terms of the twist matrix γ_k. The trace is in the bifundamental or fundamental representation of the gauge group, for the A and M surfaces, respectively. As previously discussed, the composite field-strength G is closely associated to the curvature two-form of the normal bundle $\mathcal{N}_k$. More precisely, $x_i = iG_i/\pi$, and by plugging in the relations (5.2) above, one gets after some simple algebra

$$\mathrm{index}(D_{g^k}) = -i \int_{R^{3,1}} C_k \widehat{A}_k(G)\,\mathrm{ch}_k(F) \widehat{A}(R)\,, \tag{5.3}$$

which corresponds to the k-th term in the partition functions (4.16) on A and M.

The case of the G-index of the signature complex can be treated similarly. The G-signature index theorem is

$$\mathrm{index}(\mathcal{D}^+_{g^k}) = \int_{X^G} \frac{\mathrm{ch}(\mathcal{T}^+_k - \mathcal{T}^-_k)\,\mathrm{ch}_k(\mathcal{N}^+_k - \mathcal{N}^-_k)}{\mathrm{ch}_k(\tilde{\mathcal{N}}_k) e(\mathcal{T}_k)}\,\mathrm{Td}(\mathcal{T}^C_k)\,, \tag{5.4}$$

where $\mathcal{T}^\pm_k = {}^\pm \wedge \mathcal{T}^*_k$, $\mathcal{N}^\pm_k = {}^\pm \wedge \mathcal{N}^*_k$, in terms of the cotangent and conormal bundles $\mathcal{T}^*_k$ and $\mathcal{N}^*_k$. More explicitly, we have

$$\begin{aligned}
\mathrm{ch}(\mathcal{T}^+_k - \mathcal{T}^-_k) &= \prod_{a=1}^{2} (e^{x_a} - e^{-x_a})\,, \\
\mathrm{ch}_k(\mathcal{N}^+_k - \mathcal{N}^-_k) &= \prod_{i=1}^{3} (e^{-x_i} e^{-2i\pi k v_i} - e^{x_i} e^{2i\pi k v_i})\,.
\end{aligned} \tag{5.5}$$

Similarly to the previous case, the index can then be written as

$$\mathrm{index}(\mathcal{D}^+_{g^k}) = -i \int_{R^{3,1}} C_{2k} \widehat{L}_k(G) \widehat{L}(R)\,, \tag{5.6}$$

which corresponds to the k-th term in the partition functions (4.16) on K.

6. Factorization

Having computed all the four amplitudes contributing to the anomaly, we are now in the position of facing the interpretation in terms of quantum anomalies and classical inflows, and understand the mechanism allowing their cancellation. We will also extract all the anomalous couplings to twisted RR fields by factorization.

6.1 Quantum anomalies

The anomaly arising from open string states is given by the A and M partition functions: $\mathcal{A}_{\rm open} = \mathcal{A}_A + \mathcal{A}_M$. In total, one has:

$$\mathcal{A}_{\rm open} = \frac{i}{4N} \sum_{k=1}^{N-1} C_k A_k(G) \Big[\mathrm{ch}^2_k(F) - \mathrm{ch}_{2k}(2F) \Big] \widehat{A}(R)\,. \tag{6.1}$$

The anomaly from closed string states comes instead from the K and T partition functions: $\mathcal{A}_{\rm closed} = \mathcal{A}_K + \mathcal{A}_T$, where $\mathcal{A}_T$ denotes all the contributions in (4.24). It turns out that the $\mathcal{A}_K$ precisely cancels against the untwisted RR part of $\mathcal{A}_T$. This reflects the fact that all the descendants of the anti-self-dual 4-form of the original type-IIB theory are projected out by the Ω-projection. One is then left with:

$$\begin{aligned}
\mathcal{A}_{\rm closed} = & \frac{i}{2N} \sum_{k=1}^{N-1} C_k \Big[\widehat{A}_k(G) \widehat{G}(R) + \widehat{G}_k(G) \widehat{A}(R) \Big] + \\
& + \frac{1}{2N} \sum_{k=0}^{N-1} \sum_{l=1}^{N-1} N_{k,l}\, \mathrm{ch}(G) \widehat{A}(R)\,.
\end{aligned} \tag{6.2}$$

The quantum anomalies (6.1) and (6.2) can be qualitatively understood in their alternative interpretation as anomalies involving internal reparametrizations. Indeed, in that context it is easy to discuss the representation of each state under all the symmetries. In particular, all the open string states and the untwisted closed string states transforms under tangent and internal reparametrizations in a way which is dictated essentially by dimensional reduction. This is easily made precise after recalling that the characteristic classes (4.4), (4.5) and (4.6) signal spinor, gravitino and self-dual representations under tangent reparametrizations, and similarly (4.7), (4.8) and (4.9) correspond to spinor, gravitino and self-dual representations under internal reparametrizations. The open string contribution (6.1) comes clearly from a chiral spinor in $D = 10$, which once dimensionally reduced to $D = 4$ gives rise to a multiplet of chiral spinors transforming as an internal spinor. Similarly, the untwisted part (first two terms) of the closed string contribution (6.2) come from a chiral gravitino in $D = 10$, which when dimensionally reduced to $D = 4$ gives rise to a multiplet of chiral gravitinos transforming as an internal spinor (first term), plus a multiplet of chiral spinors transforming as an internal gravitino (second term). Even the canceled contribution of the states projected out by the orientifold projection in summing the K and T surfaces can be understood. They come, as anticipated, from a self-dual form in $D = 10$, which is eventually projected out, but would give rise in $D = 4$ to a multiplet of self-dual forms transforming as an internal self-dual from. The only contribution which cannot be understood in this way is the twisted part (third term) of (6.2). It is clear that the correponding states must be chiral spinors, and one can argue intuitively that they should transform in a simpler way than untwisted fields under internal reparametrizations (not as tensors), since they arise at given fixed-points in the internal space. Indeed, it is clear from the Chern character in their contribution that they transform with a common U(1) charge.

The interpretation and analysis of (6.1) and (6.2) as sigma-model anomalies is postponed to section 7.

6.2 Classical inflows

The GS inflow, which cancels the anomalies computed in previous section, is given by the $t \to 0$ limit of the A, M and K partition functions (4.16). By factorization, it is then possible to obtain the anomalous couplings responsible for the inflows.

As in the case without composite background [26, 27], the A, M and K partition functions have to factorize exactly. This is made possible by the following non-trivial identities among the characteristic classes defined in section 4:

$$\sqrt{\widehat{A}(R)}\sqrt{\widehat{L}\left(\frac{R}{4}\right)} = \widehat{A}\left(\frac{R}{2}\right), \tag{6.3}$$

$$\sqrt{\widehat{A}_{2k}(G)}\sqrt{\widehat{L}_k\left(\frac{G}{4}\right)} = \widehat{A}_k\left(\frac{G}{2}\right). \tag{6.4}$$

Indeed, by performing suitable rescalings (allowed by the fact that only the 6-form component of all the polynomials is relevant) and summing the k-th and the $N-k$-th terms in the sums since they correspond to the same closed string twisted sector, the partition functions (4.16) can be rewritten in the factorized form

$$
\begin{aligned}
Z_A &= \frac{i}{2} \sum_{k=1}^{(N-1)/2} N_k Y_{(k)} \wedge Y_{(k)} \,, \\
Z_M &= i \sum_{k=1}^{(N-1)/2} N_k Y_{(2k)} \wedge Z_{(2k)} \,, \\
Z_K &= \frac{i}{2} \sum_{k=1}^{(N-1)/2} N_k Z_{(2k)} \wedge Z_{(2k)} \,,
\end{aligned} \tag{6.5}
$$

where $N_k = C_k^2$ is the number of fixed-points and

$$
\begin{aligned}
Y_{(k)} &= \frac{\epsilon_k}{\sqrt{N}} \sqrt{\left| \frac{1}{C_k} \right|} \, \mathrm{ch}_k(\epsilon_k F) \sqrt{\widehat{A}_k(\epsilon_k G)} \sqrt{\widehat{A}(R)} \,, \\
Z_{(2k)} &= -\frac{4\epsilon_k}{\sqrt{N}} \sqrt{\left| \frac{C_{2k}}{C_k^2} \right|} \sqrt{\widehat{L}_k \left(\epsilon_{2k} \frac{G}{4} \right)} \sqrt{\widehat{L} \left(\frac{R}{4} \right)} \,.
\end{aligned} \tag{6.6}
$$

This implies the following anomalous couplings [27]:

$$
S_D = \sqrt{2\pi} \sum_{k=1}^{(N-1)/2} \sum_{i_k=1}^{N_k} \int C_{(k)}^{i_k} \wedge Y_{(k)} \,, \tag{6.7}
$$

$$
S_F = \sqrt{2\pi} \sum_{k=1}^{(N-1)/2} \sum_{i_k=1}^{N_k} \int C_{(2k)}^{i_k} \wedge Z_{(2k)} \,. \tag{6.8}
$$

In these couplings, $C_{(k)}^{i_k}$ denotes the sum of all the RR forms in the k-twisted sector and at the fixed-point i_k; it contains a 4-form plus a 2-form $\tilde{\chi}_{(k)}^{i_k}$ and its dual 0-form $\chi_{(k)}^{i_k}$. The relevant components of the charges (6.6) are therefore the 0, 2 and 4-forms.

Thanks to the tadpole condition (4.2), all irreducible terms in the inflow (6.5) vanish, and no unphysical negative RR forms propagate in the transverse channel.

The total GS couplings can be obtained by summing the D-brane and fixed point contributions (6.7) and (6.8), after sending k into $2k$ in (6.7). This is allowed for N odd, and also in agreement with the fact that in the transverse channel one finds k-twisted states on A, and $2k$ twisted states on M and K for the k-th term in the partition function; in order to add the two consistently one is therefore led to the above substitution. Defining the quantities $X_{(2k)} = Y_{(2k)} + Z_{(2k)}$, one has

$$
S_{\mathrm{GS}} = \sqrt{2\pi} \sum_{k=1}^{(N-1)/2} \sum_{i_k=1}^{N_k} \int C_{(2k)}^{i_k} \wedge X_{(2k)} \,. \tag{6.9}
$$

Using the explicit form (6.6) of the charges and the tadpole cancellation condition (4.2), one can check that the total charges $X^{(0)}_{(2k)}$ with respect to the RR 4-forms are zero, and the following results for the total charges $X^{(2)}_{(2k)}$ and $X^{(4)}_{(2k)}$ with respect to the RR 2-forms $\tilde{\chi}^{i_k}_{(2k)}$ and the RR 0-forms $\chi^{i_k}_{(2k)}$ are found:

$$X^{(2)}_{(2k)} = \frac{N_k^{-1/4}}{\sqrt{N}(2\pi)} \left\{ i\,\mathrm{tr}(\gamma_{2k}F) + \frac{1}{2}\,\mathrm{tr}(\gamma_{2k}) \sum_{i=1}^{3} \tan(\pi k v_i) G_i \right\}, \tag{6.10}$$

$$\begin{aligned} X^{(4)}_{(2k)} = -\frac{\epsilon_{2k} N_k^{-1/4}}{2\sqrt{N}(2\pi)^2} \Bigg\{ & \mathrm{tr}(\gamma_{2k}F^2) - \frac{1}{32}\,\mathrm{tr}(\gamma_{2k})\,\mathrm{tr}\,R^2 + i\,\mathrm{tr}(\gamma_{2k}F) \sum_{i=1}^{3} \cot(2\pi k v_i) G_i - \\ & - \frac{1}{4}\,\mathrm{tr}(\gamma_{2k}) \Bigg[\sum_{i=1}^{3} \tan^2(\pi k v_i)(G_i)^2 + \\ & + 2 \sum_{i\neq j=1}^{3} \frac{\cos(2\pi k v_i)\cos(2\pi k v_j) - 1}{\sin(2\pi k v_i)\sin(2\pi k v_j)} G_i G_j \Bigg] \Bigg\}. \end{aligned} \tag{6.11}$$

Finally, one arrives at a very simple factorized expression for the 6-form encoding the complete sigma-gauge-gravitational anomaly and its opposite inflow:

$$\mathcal{A}^{(6)} = \mathcal{I}^{(6)} = i \sum_{k=1}^{(N-1)/2} N_k X^{(2)}_{(2k)} \wedge X^{(4)}_{(2k)}\,. \tag{6.12}$$

7. Field theory outlook

In this section, we shall address the interpretation of the results found through the string computation within the low-energy supergravity. The 2-form couplings (6.10) will be responsible for a modification of the kinetic terms of the twisted RR axions, and will force the latter to transform non-homogeneously under gauge and modular transformations. The 4-form couplings will then become anomalous and generate the GS inflow required to cancel all the anomalies.

In the following, we focus on FFG_i and RRG_i anomalies, since these can be compared to field theory expectations.

FFG_i anomalies. This kind of anomalies arise only from open string states. To get an explicit expression from the expansion of (6.1), it is convenient to transform k into $2k$ in the annulus contribution. Using (4.2), one finds for the non-abelian part:

$$\mathcal{A}^{FFG_i} = -\frac{i}{2N(2\pi)^3} \sum_{k=1}^{N-1} C_k \tan(\pi k v_i)\,\mathrm{tr}(\gamma_{2k}F^2) G_i\,. \tag{7.1}$$

As we have seen in section 2, FFG_i anomalies are encoded in some coefficients b^i_a defined through

$$\mathcal{A}^{FFG_i} = \frac{i}{2(2\pi)^3} b^i_a\,\mathrm{tr}(F_a^2) G_i\,, \tag{7.2}$$

where the index a label the various factors of the gauge group. The coefficients b_a^i are found to be in agreement with those computed in [20] for any a, i and for both the $\mathbb{Z}_3$ and $\mathbb{Z}_7$ models. This confirms the conjectured anomaly cancellation of mixed FFG_i anomalies through a GS mechanism involving RR axions, as proposed in [20]. Indeed, the same anomaly polynomial is reproduced and by factorization the expected couplings are obtained, i.e. the second term in (6.10). The one-forms $(X^{(2)}_{(2k)})^{(0)}$ modify the kinetic terms for the axions $\chi^{i_k}_{(2k)}$. Strictly speaking it is only the combination

$$\chi_{2k} = \frac{1}{\sqrt{N_k}} \sum_{i_k=1}^{N_k} \chi^{i_k}_{(2k)} \tag{7.3}$$

that gets modified, since all the axions enter in a completely symmetric way in the GS mechanism [27]. Whereas the first term in (6.10) induces a non-homogeneous U(1) transformation for χ_{2k} that eventually leads to a Higgs mechanism through which χ_{2k} itself is eaten by the U(1) field, the second term in (6.10) leads to a non-homogeneous modular transformation for χ_{2k}. Note that the WZ descent for G_i is $G_i^{(1)} = -i/2[\lambda^i(t^i) - \bar{\lambda}^i(\bar{t}^i)]$, with $\lambda^i(t^i)$ the lowest component of (2.10). Correspondingly, the (normalized) kinetic term for χ_{2k} will be invariant under sigma-model transformations if the associated superfields M_{2k} transforms, under $\mathrm{SL}(2,\mathbb{R})^i$, in the following non-homogeneous way:

$$M_{2k} \longrightarrow M_{2k} - \frac{1}{8\pi^2} \alpha^i_{2k} \lambda^i(T^i)\,, \tag{7.4}$$

with

$$\alpha^i_{2k} = \frac{(2\pi)^{3/2}}{\sqrt{N}} N_k^{1/4}\, \mathrm{tr}(\gamma_{2k}) \tan(\pi k v_i)\,. \tag{7.5}$$

RRG_i **anomalies.** This kind of anomalies gets contribution both from open and closed string states. Performing the same manipulation as before in the sum over k for the annulus contribution, and summing the contributions (6.1) and (6.2) from open and closed strings, one finds

$$\begin{aligned}\mathcal{A}^{RRG_i} = \frac{i}{96N(2\pi)^3} \Bigg\{ &\sum_{k=1}^{N-1} C_k \left[\tan(\pi k v_i) - \frac{1}{2}\cot(\pi k v_i)\right] \mathrm{tr}(\gamma_{2k}) + \\ &+ \sum_{k=1}^{N-1} C_k \cot(\pi k v_i) \Bigg[21 + 1 - 2 \Bigg(4\sin^2(\pi k v_i) + \\ &\qquad + \sum_{j=1}^{3} \cos(2\pi k v_j) \Bigg) \Bigg] + \\ &+ \sum_{k=0}^{N-1} \sum_{l=1}^{N-1} N_{k,l} \Bigg\} \mathrm{tr}\, R^2 G_i\,. \end{aligned} \tag{7.6}$$

The first line comes from the open strings, and the second and third line from untwisted and twisted closed strings. As expected the untwisted RNS contribution in the first line encodes the anomaly of the gravitino (21), the dilatino (1), and the fermionic partners of the three untwisted moduli ($-2(4\sin^2(\pi k v_i)+\sum_j \cos(2\pi k v_j))$). The RNS twisted sector contribution in the second line corresponds instead to the anomaly of the neutralini. By explicit evaluation one finds finally:

$$\mathcal{A}^{RRG_i} = -\frac{i}{48(2\pi)^3}\begin{bmatrix} -10+21+1-3-27 \\ -6+21+1-1-21 \end{bmatrix} \operatorname{tr} R^2 G_i\,. \tag{7.7}$$

The coefficient in the square brackets has to be compared with $b^i_{\text{grav.}} = b^i_{\text{open}} + b^i_{\text{closed}}$ of [22], the upper and lower raws corresponding to the $\mathbb{Z}_3$ and $\mathbb{Z}_7$ models, respectively. In the notation of [20], the explicit form of these coefficients is:

$$\begin{aligned} b^i_{\text{closed}} &= 21 + 1 + \delta^i_T + \sum_\alpha (1 + 2n^\alpha_i)\,, \\ b^i_{\text{open}} &= -\dim G + \sum_{a=1}^{3} (1 + 2n^a_i)\eta_a\,. \end{aligned} \tag{7.8}$$

In b^i_{closed}, δ^i_T is the total contribution of the untwisted moduli (nine for the $\mathbb{Z}_3$ model and three for the $\mathbb{Z}_7$ model) and α runs over all the twisted massless states. These are assumed to have modular weight n^α_i as defined in section 2, and in [22] it was assumed that $n^\alpha_i = 0$. In b^i_{open}, the first term is the contribution of the gaugini, where G is the total gauge group of the model, $n^i_a = -\delta^i_a$ are the modular weights of the charged fields C^a and η_a simply counts the number of charged states belonging to the group a. Comparing the string result (7.7) with the field theory expectations given by (7.8), one finds agreement for b^i_{open} (first number in (7.7)) and for the untwisted contribution in b^i_{closed} (next three numbers), but opposite signs for the twisted contribution (last number). Assuming the validity of (7.8), agreement with the string results would predict twisted modular weights $n^\alpha_i = -1$, $\forall i, \alpha$. This is in apparent contradiction with the non-homogeneous transformation (7.4) required for the cancellation of sigma-gauge anomalies.

The sign that we find for the contribution of twisted modulini is crucial for the realization of the GS anomaly cancellation mechanism, since it directly influences the factorizability of the quantum anomaly. We do not have a full understanding of this discrepancy; rather, we would like to revisit the assumptions at the origin of the above field theory analysis and point out a few delicate points. A first point to observe, in comparing string results with field theory expectations in $D = 4$ $N = 1$ models, is that the first are believed to be expressed in terms of linear multiplets, whereas the latter are often given in terms of the usual chiral multiplets, as is the case for (7.8). The two multiplets are related by the so called linear multiplet - chiral multiplet duality, that is basically the extension to superfields of the duality between a two-form and a scalar in four dimensions. It is also known that the GS terms modify the above duality [21]. Correspondingly, particular attention has to be paid in comparing the results (7.7) with field theory formulae obtained using the chiral multiplet basis as (7.8) (see for instance footnote 7). A second very important point is that the expression (7.8) for the anomaly coefficients are valid only under the assumption that the Kähler potential $K^{(M)}$ for twisted fields and their modular transformations have the form (2.11). Unfortunately, the potential $K^{(M)}$ has not been computed yet in type-IIB orientifold models, and therefore it is not possible to verify directly these assumptions.

We propose that the Kähler potential for twisted fields does in fact not satisfy the assumptions at the origin of (7.8), so that the whole field-theory derivation of sigma-model anomalies, as reviewed in section 2 and expressed in (7.8), has to be revisited [29]. A first possibility is that $K^{(M)} \sim (M + \bar{M})^2$, as proposed in [28]. This potential satisfies the assumptions behind (7.8) (and leads to $n_i^\alpha = 0$ as assumed in [22]), but only if one neglects the correction induced by the GS couplings (6.10) and (6.11). These are indeed present, as described in [45], and it might be that they must be considered on equal footing with the rest of the potential.[12] A second possibility is that $K^{(M)}$ is a different function of the twisted moduli, invariant under sigma-model transformations and the shift (7.4), whose form does not satisfy the assumptions leading to (7.8).

An explicit string computation of $K^{(M)}$ would therefore be extremely interesting and could give a definite answer to the problems raised above. Unfortunately, such a computation appears to be quite complicated.

8. Conclusions

In this paper, we have studied along the lines of [26, 27] the pattern of sigma-gauge-gravitational anomaly cancellation in compact type-IIB $D = 4$ $N = 1$ $\mathbb{Z}_N$ orientifolds with N odd. Our main result is that all the anomalies are cancelled through a generalized GS mechanism.

The starting point of our analysis is the definition of the effective vertex operator corresponding to the sigma-model connection. We provided several general arguments for indentifying it with the vertex encoding Kähler deformations of the orbifold, but we were able to give only a not completely rigorous derivation which cannot be taken as a proof. A posteriori, this identification is strongly supported also by the results obtained for the anomalies using this vertex. Under the assumption that the effective vertex is indeed correct, we generalize the known results [46, 27] for gauge-gravitational anomalies and show that all possible sigma-gauge-gravitational

[12] This seems quite strange from a string theory point of view, but we believe it might be reasonable in light of the string coupling dependence of the definition (3.5) for the T^i moduli.

anomalies are cancelled through a GS mechanism. This is essentially what was proposed in [20] for sigma-gauge anomalies,, and seems to evade the arguments of [22] against a field theory mechanism for the cancellation of sigma-gravitational anomalies. We interpret this discrepancy as evidence that the comparison of the string results with the field theory expectations is probably more subtle than expected. In particular, we propose that the actual Kähler potential for twisted fields does not satisfy the usual assumptions made in the literature, so that the interpretation of our string results remains actually open.

We would like to stress that the present results imply a *full* cancellation of anomalies in all possible channels. The torus contribution presents a surprising cancellation and yields vanishing anomalies and inflows. This implies in particular that the dilaton field does not play any role in the GS mechanism. The annulus, Möbius strip and Klein bottle contributions are instead topological, guaranteeing an exact cancellation between quantum anomalies and classical inflows mediated by twisted RR axions.

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A. ϑ-functions

For convenience, we introduce here a convenient notation for the twisted θ-functions appearing in orbifold and orientifold partition functions. In particular, in order to keep manifest the origin of each of these, we shall define

$$\theta_1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) = \theta \begin{bmatrix} \frac{1}{2}+\alpha \\ \frac{1}{2}+\beta \end{bmatrix} (z|\tau) \,, \tag{A.1}$$

$$\theta_2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) = \theta \begin{bmatrix} \frac{1}{2}+\alpha \\ 0+\beta \end{bmatrix} (z|\tau) \,, \tag{A.2}$$

$$\theta_3 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) = \theta \begin{bmatrix} 0+\alpha \\ 0+\beta \end{bmatrix} (z|\tau) \,, \tag{A.3}$$

$$\theta_4 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) = \theta \begin{bmatrix} 0+\alpha \\ \frac{1}{2}+\beta \end{bmatrix} (z|\tau) \,, \tag{A.4}$$

in terms of the usual twisted θ-functions

$$\theta \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (z|\tau) = \sum_n q^{\frac{1}{2}(n-\alpha)^2} e^{2\pi i(z-\beta)(n-\alpha)} \,. \tag{A.5}$$

All the properties and identities relevant to the usual θ-functions (A.5) easily translate into analogous properties of (A.1)–(A.4).

B. Anomalies in type-IIB string theory

The cancellation of gravitational anomalies in type-IIB supergravity theories requires non-trivial identities involving the anomalies of dilatinos, gravitinos and self-dual forms. These are given by

$$I_{1/2} = \widehat{A}(R)\,, \qquad I_{3/2} = \widehat{G}(R)\,, \qquad I_A = -\frac{1}{8}\widehat{L}(R)\,, \tag{B.1}$$

in terms of the characteristic classes (4.4)–(4.6). From a type-IIB string theory point of view, anomaly freedom is more manifest since the corresponding torus amplitude is perfectly finite. However, it is clear that in the low-energy field theory limit one has to reproduce in string theory the same non-trivial identity. This can be regarded as a technique to compute anomalous Feynman diagrams using a string regularization. Due to the relevance of the torus amplitude in the mixed sigma-gauge-gravitational anomalies considered in this paper, we find useful to report here some details on how to reproduce in type-IIB string theory the aforementioned identity.

As explained in sections 3 and 4, the only potentially anomalous contributions on the torus come from the three odd-even and the three even-odd spin-structures, and the total anomaly is given by the expression (4.17), in terms of the partition functions (4.18) defined through the deformation vertices (4.20) and (4.21). It turns out that the two partition functions (4.18) will always be modular invariant, so that the $\partial\mathcal{F}_0$ component of the boundary gives a vanishing contribution. Moreover, on the other component $\partial\mathcal{F}_\infty$ of the boundary, the odd-even and even-odd partition functions become equal and sum. In the following, we will therefore restrict to the odd-even spin-structures.

$D = 10$. In the ten-dimensional case, the partition functions (4.18) are particularly easy to compute. One gets

$$Z_T^{S_\alpha} = \frac{1}{4}\prod_{a=1}^{5}\left[\frac{ix_a}{\theta_1(ix_a/\pi|\tau)}\theta_\alpha(ix_a/\pi|\tau)\right]\frac{\eta^3(\tau)}{\theta_\alpha(0|\tau)}\,. \tag{B.2}$$

Here $\alpha = 2,3,4$ represent respectively the RR, RNS_+ and RNS_- spin-structures, the factor of $1/4$ is due to the left and right GSO projections and $x_a = \lambda_a/2\pi$, in terms of the skew eigenvalues λ_a of the gravitational curvature R. The first fraction is the contribution of the bosonic and fermionic fields, whereas the last fraction is due to ghosts and superghosts. Taking the limit $\tau_2 \to \infty$, one obtains in the RR spin-structure

$$Z_T^{RR} \longrightarrow \frac{1}{8}\prod_{a=1}^{5}\frac{x_a}{\tanh x_a} \,. \tag{B.3}$$

In the RNS$_\pm$ spin-structures, similarly

$$Z_T^{RNS} = Z_T^{RNS_+} - Z_T^{RNS_-} \longrightarrow \prod_{a=1}^{5}\frac{x_a/2}{\sinh x_a/2}\left(2\sum_{b=1}^{5}\cosh x_b - 2\right), \tag{B.4}$$

where we rescaled by a factor of 2 the x_a's, exploiting the fact that only the 12-form of (B.4) is relevant. Notice that the leading "tachyonic" terms in $Z_T^{RNS_\pm}$ cancel in the combination $Z_T^{RNS_+} - Z_T^{RNS_-}$. By summing the three contributions one finds as expected the anomaly of an anti-chiral gravitino and of a chiral dilatino from the RNS/NSR sector and that of an (anti)self-dual tensor from the RR sector. In total, one gets

$$I_T = -I_{3/2} + I_{1/2} - I_A = 0 \,, \tag{B.5}$$

ensuring the absence of pure gravitational anomalies in $D = 10$ type-IIB supergravity and superstring theory [47].

$D = 6$ **on** $T^4/\mathbb{Z}_N$**.** As usual in orbifold theories, the partition functions (4.18) contain a sum over orbifold twisted sectors l, as well as a projection on $\mathbb{Z}_N$-invariant states; see (4.19). In the following, we will further distinguish between the contributions coming from untwisted and twisted sectors. The twist vector is $v_i = (1/N, -1/N)$, $C_k = \prod_i (2\sin(\pi k v_i))$, and $N_{k,l}$ are the number of points that are at the same time k and l-fixed. The total partition function is

$$Z_T = \sum_\alpha (-)^\alpha \sum_{l=0}^{N-1} Z_T^{S_\alpha(l)} \,, \tag{B.6}$$

where

$$\begin{aligned} Z_T^{S_\alpha(l)} &= \frac{1}{4N}\sum_{k=0}^{N-1} N_{k,l}\prod_{i=1}^{2}\frac{\theta_\alpha\begin{bmatrix} lv_i \\ kv_i\end{bmatrix}(0|\tau)}{\theta_1\begin{bmatrix} lv_i \\ kv_i\end{bmatrix}(0|\tau)} \times \\ &\quad \times \prod_{a=1}^{3}\left[\frac{ix_a}{\theta_1(ix_a/\pi|\tau)}\theta_\alpha(ix_a/\pi|\tau)\right]\frac{\eta^3(\tau)}{\theta_\alpha(0|\tau)} \,. \end{aligned} \tag{B.7}$$

In the $\tau_2 \to \infty$ limit, one finds in the RR spin-structures:

$$Z_T^{RR(0)} \longrightarrow \frac{1}{8N}\sum_{k=0}^{N-1} C_{2k}\prod_{a=1}^{3}\frac{x_a}{\tanh x_a}\,,$$
$$Z_T^{RR(l\neq 0)} \longrightarrow \frac{1}{8N}\sum_{k=0}^{N-1} N_{k,l}\prod_{a=1}^{3}\frac{x_a}{\tanh x_a}\,. \tag{B.8}$$

One can easily check that for any $N = 2, 3, 4, 6$, the total is given by

$$Z_T^{RR} \longrightarrow 2\prod_{a=1}^{3}\frac{x_a}{\tanh x_a}\,. \tag{B.9}$$

In the RNS$_\pm$ spin-structures one has to pay particular attention in taking the limit, because when $l = N/2$, the fields in the internal directions have zero modes. One finds the following results:

$$Z_T^{RNS_\pm(0)} \longrightarrow \pm\frac{1}{2N}\sum_{k=0}^{N-1} C_k\prod_{a=1}^{3}\frac{x_a/2}{\sinh x_a/2}\left(2\sum_{b=1}^{3}\cosh x_b - 2 + \sum_{i=1}^{2}(2\cos 2\pi k v_i)\right),$$
$$Z_T^{RNS_\pm(l\neq 0,N/2)} \longrightarrow \pm\frac{1}{N}\sum_{k=0}^{N-1} N_{k,l}\prod_{a=1}^{3}\frac{x_a/2}{\sinh x_a/2}\,,$$
$$Z_T^{RNS_+(N/2)} \longrightarrow \frac{1}{2N}\sum_{k=0}^{N-1} N_{k,N/2}\prod_{a=1}^{3}\frac{x_a/2}{\sinh x_a/2}\prod_{i=1,2}(2\cos\pi k v_i)^2$$
$$Z_T^{RNS_-(N/2)} \longrightarrow -\frac{1}{2N}\sum_{k=0}^{N-1} N_{k,N/2}\prod_{a=1}^{3}\frac{x_a/2}{\sinh x_a/2}\prod_{i=1,2}(2\sin\pi k v_i)^2\,. \tag{B.10}$$

We omitted the leading "tachyonic" term that, as in the previous case, will cancel in taking the sum $Z_T^{RNS_+} - Z_T^{RNS_-}$. One can easily verify that the total result in the RNS sectors, obtained by summing over the two RNS$_\pm$ contributions and over all twisted and untwisted sectors, is the same for any $N = 2, 3, 4, 6$ and given by

$$Z_T^{RNS} = -2\prod_{a=1}^{3}\frac{x_a/2}{\sinh x_a/2}\left(2\sum_{b=1}^{3}\cosh x_b - 22\right). \tag{B.11}$$

Putting all together, one gets finally

$$I_T = 2(I_{3/2} - 21 I_{1/2} - 8 I_A) = 0 \tag{B.12}$$

ensuring the absence of purely gravitational anomalies in type-IIB theory on $T^4/\mathbb{Z}_N$.

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C. Vanishing of the torus amplitude

We show here that the whole 6-form component of the torus amplitude, including G^3 anomalies, vanishes. For the RRG_i terms, one gets

$$\begin{aligned}
Z_T^{R^2G} &= \frac{i}{96N(2\pi)^3}\sum_{i=1}^{3}\left\{-4\sum_{k=1}^{N-1}C_{2k}\sin^{-1}(2\pi kv_i)+\sum_{k=1}^{N-1}C_k\cot(\pi kv_i)\times\right.\\
&\qquad\times\left[21+1-2\left(4\sin^2(\pi kv_i)+\sum_{j=1}^{3}\cos(2\pi kv_j)\right)\right]+\\
&\qquad\left.+\sum_{k=0}^{N-1}\sum_{l=1}^{N-1}N_{k,l}\right\}\mathrm{tr}\,R^2G_i\\
&= -\frac{i}{48(2\pi)^3}\begin{bmatrix}8+21+1-3-27\\ 0+21+1-1-21\end{bmatrix}\mathrm{tr}\,R^2\left(\sum_{i=1}^{3}G_i\right)=0\,,
\end{aligned}\tag{C.1}$$

where we reported in square bracket the explicit values for both the $\mathbb{Z}_3$ (up) and $\mathbb{Z}_7$ (down) orientifolds.

Consider next the $G_iG_jG_p$ terms. The RR twisted contributions vanish as before, whereas the RNS twisted ones are present and can be easily read from the last line of (4.24). On the contrary, the untwisted RR and RNS contributions requires more work. However, one can now put to zero the gravitational curvature. By doing so, the contribution of the superghosts cancels that of one of the two complex spacetime fermions in (4.22) and one can therefore use the Riemann identity to simplify the result. In the $\tau_2\to\infty$ limit, one has then:

$$\begin{aligned}
&\frac{8\prod_i\cos(\pi kv_i+G_i/2\pi)-2\sum_i\cos 2(\pi kv_i+G_i/2\pi)-2}{8\prod_i\sin(\pi kv_i+G_i/2\pi)}=\\
&\qquad=-2\sin\left(\sum_{p=1}^{3}\frac{G_p}{4\pi}\right)\prod_{i=1}^{3}\frac{\sin\left[(\pi kv_i+G_i/2\pi)-(\sum_p G_p/4\pi)\right]}{\sin(\pi kv_i+G_i/2\pi)}\,.
\end{aligned}$$

The relevant cubic term of the partition function are now easily computed, and one finds:

$$\begin{aligned}
Z_T^{G^3} &= \frac{i}{24N(2\pi)^3}\left\{\sum_{k=1}^{N-1}N_k\left[\left(3\sum_{i=1}^{3}\prod_{j\neq i=1}^{3}\cot(\pi kv_i)-5\right)\left(\sum_{p=1}^{3}G_p\right)^3+\right.\right.\\
&\qquad\left.+6\sum_{i=1}^{3}\sin^{-2}(\pi kv_i)G_i\left(\sum_{p=1}^{3}G_p\right)^2\right]-\\
&\qquad\left.-2\sum_{k=0}^{N-1}\sum_{l=1}^{N-1}N_{k,l}\left(\sum_{p=1}^{3}G_p\right)^3\right\}\\
&= \frac{i}{4N(2\pi)^3}\begin{bmatrix}9-15+24-18\\ 3-5+16-14\end{bmatrix}\left(\sum_{p=1}^{3}G_p\right)^3=0\,.
\end{aligned}\tag{C.2}$$

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